Braiding of two spiraling laser beams due to plasma wave wakes

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We study how two Gaussian laser beams interact through plasma wave wakes produced when they co-propagate in a plasma. Using a variational principle, we derive equations of motion for the centroid of each beam, and find braided centroid solutions. These results can be generalized to other nonlinear optical media with noninstantaneous nonlinearity.

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The mutual interaction between distinct laser beams in nonlinear optical media, including between solitons, has received much recent attention [1–4]. The nonlinearities considered in these studies are all instantaneous, namely depending only on the local field intensity. Recently, it was shown that when two partially overlapping laser beams propagate in a plasma, there is an effective attractive force between the two beams due to the relativistic electron mass nonlinearity [5]. The attractive force can cause the two beam centroids to spiral around each other. However, in a plasma, the ponderomotive force of each laser can excite plasma wave wakes. The electron density modulation of the wakes provides another nonlinearity to the plasma's index of refraction. More importantly, the presence of the wake breaks the translational symmetry along the propagation direction and is an example of a noninstantaneous nonlinearity. Therefore, instead of behaving as a rigid rod described by a $(2+1)$-dimensional (D) system [2–4], a single laser's centroid can go unstable to transverse or so-called hosing oscillations and must be described by a $(3+1)$D system [6,7]. The interaction of multiple lasers can be even more interesting. In Ref. [5], fully nonlinear three-dimensional particle-in-cell simulations showed that two lasers propagating in plasmas can actually intertwine and form braided patterns. To the best of our knowledge, such behavior has never been discussed for other nonlinear optical media.

The purpose of this paper is to show that when the medium has a noninstantaneous nonlinearity, i.e., the media has memory, the solutions to the nonlinear wave equations are more rich in behavior. Motivated by the observations of our simulations [5] we concentrate on plasmas, but we believe our results should apply equally well to other media. We adopt the variational method to obtain approximate behavior, and derive equations of motion for the centroids of two lasers interacting through a noninstantaneous nonlinearity.

We begin with the weakly nonlinear coupled equations for the evolution of the lasers and plasma wave wakes [8],

$$\left(2ik_0\frac{\partial}{\partial\tau}+\nabla_\perp^2-k_p^2\right)\vec{a}=-k_p^2\phi\vec{a}, \tag{1}$$

$$\left(\frac{\partial^2}{\partial\psi^2}+k_p^2\right)\phi=\frac{k_p^2}{4}|a|^2, \tag{2}$$

where $\vec{a}$ is the envelope of a vector potential $\vec{A}$ with frequency ω_0, $e\vec{A}/mc^2=\vec{a}\exp[-ik_0c(t-z/c)]/2+\text{c.c}$ and ϕ is the normalized electrostatic potential. The speed of light frame variables $(\tau\equiv z,\psi\equiv ct-z)$ is used. We define $k_p^2\equiv 4\pi ne^2/mc^2$ with n the plasma density and e, m, c, and k_0 are the electron charge, electron mass, speed of light, and vacuum laser wave number, respectively. In the weakly nonlinear limit used here, ϕ includes both the electron density modulation and the relativistic mass effect, $\phi=-\delta n/n+a^2/4$. If the assumption of translational symmetry in ψ direction were taken for ϕ, i.e., $\partial\phi/\partial\psi\equiv 0$, then $\phi=a^2/4$ from Eq. (2) and only the relativistic mass effect would be present. Viewed in another way, the solution of Eq. (2) can be formally written as $\phi=(k_p^2/4)\int d\psi'|a(\psi')|^2G(\psi-\psi')$ where the Green's function G for Eq. (2) is $G=H(\psi-\psi')\sin[k_p(\psi-\psi')]$ (H is the Heaviside step function). Equations (1) and (2) would then reduce to the nonlinear Schrödinger equation for a cubic nonlinearity if a δ function was used for the Green's function, $G=\delta(\psi-\psi')$. When the Green's function is not a δ function, the nonlinearity becomes noninstantaneous, which is the topic of this paper.

We next assume that there are two distinct laser beams that are orthogonally polarized, $\vec{a}=\hat{x}_1a_1+\hat{x}_2a_2$ where $(\hat{x}_1,\hat{x}_2)=(\hat{x},\hat{y})$. Substituting the expression for $\vec{a}$ into Eqs. (1) and (2) gives

$$\left(2ik_0\frac{\partial}{\partial\tau}+\nabla_\perp^2-k_p^2\right)a_1=-k_p^2\phi a_1, \tag{3}$$

$$\left(2ik_0\frac{\partial}{\partial\tau}+\nabla_\perp^2-k_p^2\right)a_2=-k_p^2\phi a_2, \tag{4}$$

$$\left(\frac{\partial^2}{\partial\psi^2}+k_p^2\right)\phi=\frac{k_p^2}{4}(|a_1|^2+|a_2|^2). \tag{5}$$

We use variational principle methods to obtain approximate yet illustrative solutions to the above equations [2–5,7,9]. Extending the single laser case [7], the Lagrangian density for these equations is

$$\mathcal{L}=\sum_{j=1,2}\left[ik_0\left(a_j\frac{\partial a_j^*}{\partial\tau}-a_j^*\frac{\partial a_j}{\partial\tau}\right)+\nabla_\perp a_j^*\cdot\nabla_\perp a_j+k_p^2a_ja_j^*\times(1-\phi)\right]-2\left(\frac{\partial\phi}{\partial\psi}\right)^2+2k_p^2\phi^2. \tag{6}$$

For the trial functions for each a and ϕ, we use Gaussian functions, which characterize a field quantity by its amplitude, spot size, and centroid. Generally, the spot size evolution is only weakly coupled to the centroid evolution. (In fact for linear perturbations they are completely decoupled [7].) To isolate the centroid movement and simplify the calculation, we will use one fixed spot size in all our trial functions. (We implicitly assume that a fixed spot size can be achieved through self-focusing [10] by choosing a proper laser power.) The laser amplitudes will also be kept constant because of power conservation [5,7]. To simplify things further, we will assume that the two laser beams are initially identical (same amplitude and same spot size).

Since each laser can generate its own wake, we will write the trial function for ϕ as a sum of two Gaussian functions, each with its own amplitude and centroid. In principle, the centroids of the lasers can be different from that of ϕ. However, Duda and Mori showed for the single laser case [7] there exists a mode where ϕ virtually follows the laser so the two centroids are essentially the same. This happens in the so-called long wavelength regime when the deviation of the centroid from a straight line has a much longer wavelength than the plasma wavelength. Here, we will concentrate on a mode where each laser has the same centroid as that of the corresponding wake it generates since the calculation is much simpler and this still admits braided solutions. Therefore, the trial functions are

$$a_j=\sqrt{\frac{P}{W}}\exp[-i\vec{k}_j\cdot(\vec{x}-\vec{X}_j)]\exp[-(\vec{x}-\vec{X}_j)^2/W^2],\quad j=1,2, \tag{7}$$

$$\phi=\sum_{j=1,2}\phi_j\exp[-2(\vec{x}-\vec{X}_j)^2/W^2]. \tag{8}$$

Here, the perpendicular momentum for a, $\vec{k}_j\equiv(k_{xj},k_{yj})$, the centroids $\vec{X}_j\equiv(X_j,Y_j)$, and wake amplitudes ϕ_j are all real functions of (τ,ψ). The spot size of ϕ is chosen as $W/\sqrt{2}$ so that $\phi=(|a_1|^2+|a_2|^2)/4$ in the limit that $\partial/\partial\psi\equiv0$.

Substituting these trial functions into Eq. (6) and integrating the Lagrangian density over the xy plane, we obtain a reduced Lagrangian density

$$\begin{aligned}L\equiv&\frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\mathcal{L}\\=&\sum_{j=1,2}P\left[k_0(k_{xj}\dot{X}_j+k_{yj}\dot{Y}_j)+\left(\frac{1}{W^2}+\frac{k_{xj}^2+k_{yj}^2}{2}\right)+\frac{1}{2}k_p^2\left(1-\frac{e^{-\alpha_{j1}}\phi_1+e^{-\alpha_{j2}}\phi_2}{2}\right)\right]+\sum_{j=1,2}\left\{-\left[\phi_j^2(X_j'^2+Y_j'^2)+\frac{1}{2}\phi_j'^2W^2\right]\right.\\&\left.+\frac{1}{2}k_p^2\phi_j^2W^2\right\}-\frac{2}{W^2}e^{-\alpha}\left(\frac{1}{2}\phi_1'\phi_2'W^4+\phi_1\phi_2\{X_1'X_2'[W^2-2(X_1-X_2)^2]+Y_1'Y_2'[W^2-2(Y_1-Y_2)^2]\right.\\&-2(X_1-X_2)(Y_1-Y_2)(X_1'Y_2'+X_2'Y_1')\}+\phi_1\phi_2'W^2[X_1'(X_2-X_1)+Y_1'(Y_2-Y_1)]+\phi_1'\phi_2W^2\\&\left.\times[X_2'(X_1-X_2)+Y_2'(Y_1-Y_2)]-\frac{1}{2}k_p^2W^4\phi_1\phi_2\right).\end{aligned} \tag{9}$$

Here α_{jk} and α are the normalized distances between the lasers and the wakes, $\alpha_{jk}\equiv(\vec{X}_j-\vec{X}_k)^2/W^2$ and between the wakes, $\alpha\equiv(\vec{X}_1-\vec{X}_2)^2/W^2$, respectively. The partial derivatives $\partial/\partial\tau$ and $\partial/\partial\psi$ are represented by the dot and the prime, respectively. The variables $\vec{k}_j$ can be eliminated [7] by using their Euler-Lagrange equations $k_0\dot{\vec{X}}_j+\vec{k}_j=0$, leaving only six variables, $\vec{X}_j$ and ϕ_j, $j=1,2$.

The six resulting Euler-Lagrange equations from the reduced Lagrangian Eq. (9) are highly nonlinear and are difficult to solve in general. In the rest of this paper, we, therefore, concentrate on those solutions that have the following symmetry property:

$$\vec{X}_1=-\vec{X}_2\equiv\vec{X}/2, \tag{10}$$

$$\phi_1=\phi_2\equiv\phi. \tag{11}$$

These assumptions are justified by the apparent symmetry in Eq. (9) between the subscripts 1 and 2 and by the existence of spiraling solutions with the similar symmetry in the instantaneous nonlinearity case [5]. The equations for $\vec{X}$ and ϕ can be obtained by substituting Eqs. (10) and (11) into the six Euler-Lagrange equations from the reduced Lagrangian Eq. (9). Then, only three out of those six equations are found to be independent. Alternatively, the same three equations can be derived by first substituting the symmetry conditions

Eqs. (10) and (11) into Eq. (9) and then deriving the Euler-Lagrange equations for $\vec{X},\phi$ from the new simplified Lagrangian density. Here, we use the second method.

After the substitution of the symmetry conditions Eqs. (10) and (11), the reduced Lagrangian density further simplifies to

$$L_{sym}=\frac{2P}{W^2}-\frac{P}{4}k_0^2\dot{\vec{X}}^2+\frac{\phi^2}{2}\left[2(e^{-\alpha}+1)k_p^2W^2+(e^{-\alpha}-1)\times(\vec{X}')^2-\frac{2}{W^2}e^{-\alpha}(\vec{X}\cdot\vec{X}')^2\right]-(e^{-\alpha}+1)\phi'^2W^2+2e^{-\alpha}\phi\phi'\vec{X}\cdot\vec{X}'-\frac{k_p^2P}{2}(e^{-\alpha}+1)\phi, \quad (12)$$

where now $\alpha=(\vec{X})^2/W^2$. The Euler-Lagrange equations for $\vec{X},\phi$ are

$$e^{\alpha}k_0^2PW^4\ddot{\vec{X}}-4\phi^2(\vec{X}\cdot\vec{X}')^2\vec{X}+4W^2\phi(\vec{X}\cdot\vec{X}')(\phi\vec{X}'+2\phi'\vec{X})+2W^4(-1+e^{\alpha})\phi(\phi\vec{X}''+2\phi'\vec{X}')+4W^2\phi^2(\vec{X}\cdot\vec{X}'')\vec{X}+2W^2\phi^2(\vec{X}')^2\vec{X}+2\phi\vec{X}(-2k_p^2W^4\phi+k_p^2W^2P-2W^4\phi'')=0, \quad (13)$$

$$(e^{\alpha}+1)\left(2W^2\phi''+2k_p^2W^2\phi-\frac{k_p^2P}{2}\right)+(1-e^{\alpha})(\vec{X}')^2\phi+\frac{2}{W^2}(\vec{X}\cdot\vec{X}')^2\phi-4(\vec{X}\cdot\vec{X}')\phi'-2[(\vec{X}')^2+\vec{X}\cdot\vec{X}'']\phi=0. \quad (14)$$

These equations can be simplified further if we seek only those solutions for which the distance between the two laser centroids is constant, i.e., $\alpha\equiv$const. This means $\vec{X}\cdot\vec{X}'\equiv0$ and $\vec{X}\cdot\vec{X}''=-(\vec{X}')^2$. Under these conditions, Eqs. (13) and (14) become

$$e^{\alpha}k_0^2PW^4\ddot{\vec{X}}+2W^4(-1+e^{\alpha})\phi(\phi\vec{X}''+2\phi'\vec{X}')-2W^2\phi^2(\vec{X}')^2\vec{X}+2\phi\vec{X}(-2k_p^2W^4\phi+k_p^2W^2P-2W^4\phi'')=0, \quad (15)$$

$$(e^{\alpha}+1)\left(2W^2\phi''+2k_p^2W^2\phi-\frac{k_p^2P}{2}\right)+(1-e^{\alpha})(\vec{X}')^2\phi=0. \quad (16)$$

The projection of Eq. (15) in the $\vec{X}'$ direction yields an even simpler relation

$$e^{\alpha}k_0^2PW^4\ddot{\vec{X}}\cdot\vec{X}'+2W^4(-1+e^{\alpha})\phi\left[\frac{1}{2}\phi\frac{d}{d\psi}(\vec{X}')^2+2\phi'(\vec{X}')^2\right]=0. \quad (17)$$

We look for uniformly braiding solutions of the form

$$\vec{X}=\sqrt{\alpha}W[\hat{x}\cos(\omega\tau+\kappa\psi)+\hat{y}\sin(\omega\tau+\kappa\psi)], \quad (18)$$

where the speed of light frame frequency ω and wave number κ are all constant. Note that as $\kappa\rightarrow0$, the lasers spiral around each other as rigid rods with a rotation frequency ω. These solutions exist for an instantaneous nonlinearity. The new types of braiding solutions arise when κ is not zero. For such a solution, $\ddot{\vec{X}}\cdot\vec{X}'=0$ and $d/d\psi(\vec{X}')^2=0$. Therefore, for a uniformly braided laser, ϕ is a constant since Eq. (17) implies that $\phi'=0$. Under these conditions, ϕ can be solved for from Eq. (16),

$$\phi=\frac{P}{4W^2}\left[1-\frac{\alpha}{2}\tanh\left(\frac{\alpha}{2}\right)\left(\frac{\kappa}{k_p}\right)^2\right]^{-1}. \quad (19)$$

This solution reduces to the instantaneous nonlinearity result $\phi=P/4W^2$ when $\kappa\rightarrow0$. As the braiding wavelength gets shorter, κ increases and ϕ gets bigger. At some value of κ, ϕ exceeds unity and our weakly nonlinear solution breaks down. Roughly speaking, there is a critical wave number $\kappa=k_p/\sqrt{(\alpha/2)\sinh(\alpha/2)}$. Note that κ needs to be less than k_p in order for our assumption that the centroids for the laser and wake be equal to remain valid.

Finally, upon substituting Eqs. (18) and (19) into Eqs. (15), we obtain the dispersion relation

$$\omega^2=\Omega_0^2\frac{1-\tilde{\kappa}^2f_1(\alpha)}{[1-\tilde{\kappa}^2f_2(\alpha)]^2}, \quad (20)$$

where $\Omega_0\equiv k_p\sqrt{Pe^{-\alpha}}/(2k_0W^2)$ is the rotation frequency when there is no braiding, $\tilde{\kappa}\equiv\kappa/k_p$ is the normalized wave number, and f_1 and f_2 are two functions of α: $f_1=(e^{2\alpha}-1+2\alpha e^{\alpha})/[2(e^{\alpha}+1)]$ and $f_2=(\alpha/2)\tanh(\alpha/2)$. This dispersion relation relates ω, the rotation frequency of the beams in τ, with $\tilde{\kappa}$, the braiding wavelength in ψ.

The braiding discussed here is a 3D phenomenon and can only be described by a $(3+1)$D system with a noninstantaneous nonlinearity. While the uniformly braiding solution of Eq. (18) is not a soliton in a strict sense because we have used an approximate variational approach, it does have a constant interlaser distance and a constant wake. It will be of interest to try to find true soliton solution in media with the noninstantaneous nonlinearity. Furthermore, we have concentrated on the so-called long wavelength limit where

$\partial^2\phi/\partial\psi^2 \ll k_p^2\phi$ in Eq. (2). It will, therefore, be of interest to study the limit where the plasma wave is nearly resonantly driven. We close by noting that the parameters required to look for braided lasers in plasmas are not severe. For example, the simulation results of Ref. [5] were for a 1 μm laser with power of a couple of terawatt in a plasma density of 1.1×10^{19} cm^{-3}.

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